

# Mesoscopic description of random walks on combs

Vicenç Méndez,<sup>1</sup> Alexander Iomin,<sup>2</sup> Daniel Campos,<sup>1</sup> and Werner Horsthemke<sup>3</sup>

<sup>1</sup>*Grup de Física Estadística. Departament de Física. Facultat de Ciències. Edifici Cc. Universitat Autònoma de Barcelona, 08193 Bellaterra (Barcelona) Spain*

<sup>2</sup>*Department of Physics, Technion, Haifa, 32000, Israel*

<sup>3</sup>*Department of Chemistry, Southern Methodist University, Dallas, Texas 75275-0314, USA*

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Combs are a simple caricature of various types of natural branched structures, which belong to the category of loopless graphs and consist of a backbone and branches. We study continuous time random walks on combs and present a generic method to obtain their transport properties. The random walk along the branches may be biased, and we account for the effect of the branches by renormalizing the waiting time probability distribution function for the motion along the backbone. We analyze the overall diffusion properties along the backbone and find normal diffusion, anomalous diffusion, and stochastic localization (diffusion failure), respectively, depending on the characteristics of the continuous time random walk along the branches.

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## I. INTRODUCTION

Random walks often provide the underlying mesoscopic mechanism for transport phenomena in physics, chemistry and biology [1–3]. A wide class of random walks give rise to normal diffusion, where the mean-square displacement (MSD),  $\langle(\Delta r)^2(t)\rangle$ , grows linearly with time  $t$  for long times. In many important applications, however, the MSD behaves like  $\langle(\Delta r)^2(t)\rangle \propto t^\gamma$ , with  $\gamma \neq 1$ , and the diffusion is anomalous [1, 2]. Anomalous diffusion can be modelled by various classes of random walks [4]. We focus on the important class of continuous time random walks (CTRWs) [1, 2]. A specific feature of a CTRW is that a walker waits for a random time  $\tau$  between any two successive jumps. These waiting times are random independent variables with a probability distribution function (PDF)  $\phi(\tau)$ , and the tail of the PDF determines if the transport is diffusive ( $\gamma = 1$ ) or subdiffusive ( $\gamma < 1$ ). Heavy-tailed waiting time PDFs give rise to subdiffusion. Realistic models of the waiting time PDF have been formulated for transport in disordered materials with fractal and ramified architecture, such as porous discrete media [5] and comb and dendritic polymers [6–8], and for transport in crowded environments [9].

A simple caricature of various types of natural branched structures that belong to the category of loopless graphs is a comb model (see Fig. 1). The comb model was introduced to understand anomalous transport in percolation clusters [10–12]. Now, comb-like models are widely employed to describe various experimental applications. These models have proven useful to describe the transport along spiny dendrites [13, 14], percolation clusters with dangling bonds [11], diffusion of drugs in the circulatory system [15], energy transfer in comb polymers [6, 7] and dendritic polymers [8], diffusion in porous materials [16–18], the influence of vegetation architecture on the diffusion of insects on plant surfaces [19], and many other interdisciplinary applications. Random

walks on comb structures provide a geometrical explanation of anomalous diffusion.

More general combs have been studied recently. For example, a numerical study of the encounter problem of two walkers in branched structures shows that the topological heterogeneity of the structure can play an important role [20]. Another example is the occupation time statistics for random walkers on combs where the branches can be regarded as independent complex structures, namely fractal or other ramified branches [21]. Finally, we want to mention studies to understand the diffusion mechanism along a variety of branched systems, where scaling arguments, verified by numerical simulations, have been able to predict how the MSD grows with time [22].

Diffusion on comb structures has also been studied by macroscopic approaches, based on Fokker-Planck equations [12], which have been applied to describe diffusive properties in discrete systems, such as porous discrete media [5], infiltration of diffusing particles from one material into another [23], and superdiffusion due to the presence of inhomogeneous convection flow [24, 25]. Other macroscopic descriptions, based on renormalizing the waiting time PDF for jumps along the backbone to take into account the transport along the branches [26], have been found useful to model continuous-time-reaction-transport processes [27] and human migrations along river networks [28].

Kahng and Redner provided a mesoscopic, probabilistic description of random walks on combs, by using the successive decimation of the discrete-time Master equation to obtain a mesoscopic balance equations for the probability of the walker to be at a given node at a given time [29]. A mesoscopic description is necessary for an accurate description of the transport properties, such as the diffusion coefficient or the mean visiting time in a branch, in terms of the microscopic parameters that characterize the random walk process.

Here we obtain transport quantities within the frame-

work of the CTRW formalism. We assume that the motion along the backbone and the branches is non-Markovian and that the motion along the branches can be non-isotropic. We reduce the dynamic effect of the branches to a waiting time PDF for the motion along the backbone by using the decimation method of Kahng and Redner. The time spent by the walker between its entry into the branches and its return to the backbone for the first time is treated as a contribution to the effective waiting time at the node where the branch crosses the backbone.

The paper is organized as follows. In Sec. II we formulate the mesoscopic description of the random walk on the comb and reduce walker's motion to an effective motion along the backbone only with a renormalized waiting time PDF for the backbone nodes. Sec. III deals with the MSD of the effective backbone motion, derives the effective diffusion coefficient, and establishes the conditions for normal diffusion, anomalous diffusion, and stochastic localization (diffusion failure) [30] in terms of the number of branch nodes and the degree of bias of the motion along the branches. We provide details of the numerical calculations in Sec. ?? and summarize our results and discuss their implications in Sec. IV.

## II. MESOSCOPIC DESCRIPTION

The simplest comb model, shown in Fig. 1, is formed by a principal axis, called the backbone, which is a one-dimensional lattice with spacing  $a$ , and identical branches that cross the backbone perpendicularly at each node. The walker moves through the comb by performing jumps

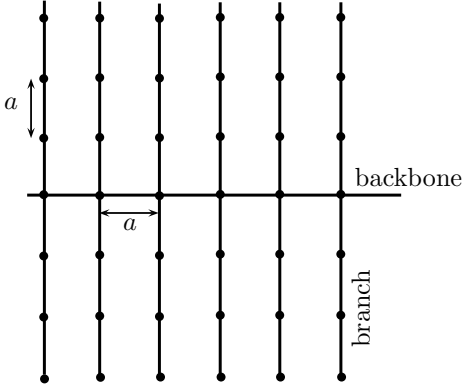


FIG. 1: Comb structure consisting of a backbone and branches. Each point represents a node where the walker may jump or wait a random time

between nearest-neighbor nodes along the backbone or along the branches. We assume that the walker performs isotropic jumps along the backbone, but the jumps along the branches may be biased, for example by an external field [10].

We derive the balance equation for the PDF  $P(x, t)$  of finding the walker at node  $x$  on the backbone at time

$t$ . When the walker arrives at a node, it waits a random time  $\tau$  before performing a new jump to the nearest node. We assume that the comb is homogeneous, and the waiting time PDF at any given node is given by  $\phi_0(\tau)$ . When the walker enters a branch, it spends some time moving inside the branch before returning to the backbone. This sojourn time can be used to determine an effective waiting time PDF  $\phi(\tau)$  for the walker's motion along the backbone. In other words, the motion of the walker on the comb can be reduced to the effective motion along a one-dimensional lattice, corresponding to the backbone only. This motion is non-Markovian and can be described mesoscopically by the Generalized Master Equation (GME)

$$\frac{\partial P}{\partial t} = \int_0^t K(t-t') dt' \times \left[ \int_{-\infty}^{\infty} P(x-x', t') \Phi(x') dx' - P(x, t') \right], \quad (2.1)$$

where  $K(t)$  is the memory kernel related to the waiting time PDF via its Laplace transform,  $K(s) = s\phi(s)/[1 - \phi(s)]$ , where  $s$  is the Laplace variable. The dispersal kernel  $\Phi(x)$  represents the probability for the walker of performing a jump of length  $x$ . If the walker moves isotropically between nearest neighbors in a one-dimensional lattice of spacing  $a$ , the dispersal kernel reads  $\Phi(x) = \delta(x-a)/2 + \delta(x+a)/2$ . We assume that the walker is initially located at  $x=0$ , i.e.,  $P(x, 0) = \delta_{x,0}$  with  $x=ia$  and  $i=0, \pm 1, \pm 2, \dots$ , where  $\delta_{x,0}$  is the Kronecker delta. Then the Laplace transform of the GME for  $x \neq 0$  reads

$$P(x, s) = \frac{\phi(s)}{2} [P(x-a, s) + P(x+a, s)]. \quad (2.2)$$

The mesoscopic balance equation for the walker on the comb being at node  $x=ia$  of the backbone is

$$P(x, s) = \frac{\phi_0(s)}{4} [P(x-a, s) + P(x+a, s)] + (1-q)\phi_0(s) [P(y=a, s) + P(y=-a, s)]. \quad (2.3)$$

Here  $P(x, s)$ ,  $P(x-a, s)$ , and  $P(x+a, s)$  is shorthand for  $P(x, y=0, s)$ ,  $P(x-a, y=0, s)$ , and  $P(x+a, y=0, s)$ , and  $P(y=a, s)$  and  $P(y=-a, s)$  stands for  $P(x, y=a, s)$  and  $P(x, y=-a, s)$ . The term  $\phi_0(s) [P(x-a, s) + P(x+a, s)]/4$  corresponds to the contribution of the walker arriving at node  $x=ia$  from the left or from the right with probability  $1/4$  after waiting a random time  $\tau$  with PDF  $\phi_0(\tau)$  at nodes  $x+a$  or  $x-a$ . As shown in Fig. 2, the walker located at the  $i$ th node of the backbone may jump to the right, left, up or down with probability  $1/4$ . We assume that the walker moves forward (away from the backbone) along the branches with probability  $q$  and back to the backbone with probability  $1-q$ . The term

$$(1-q)\phi_0(s) [P(y=a, s) + P(y=-a, s)] \quad (2.4)$$

in (2.3) corresponds the contribution of the walker arriving at the backbone node  $x$  from the first node of the upper or lower branch after waiting there a random time  $\tau$  with PDF  $\phi_0(\tau)$ .

Consider the motion along the upper branches. The lower branch dynamics is the same due to the symmetry of the comb. The mesoscopic balance equation for the first node of the upper branches reads

$$P(y = a, s) = \frac{\phi_0(s)}{4}P(x, s) + \phi_0(s)(1 - q)P(y = 2a, s). \quad (2.5)$$

The first term  $\phi_0(s)P(x, s)/4$  corresponds to the contribution of the walker arriving from the backbone, while  $\phi_0(s)(1 - q)P(y = 2a, s)$  is the contribution of the walker jumping from the upper node  $y = 2a$  to  $y = a$  with probability  $1 - q$  after waiting a random time  $\tau$  with PDF  $\phi_0(\tau)$ . Analogously, we have for the lower branches

$$P(y = -a, s) = \frac{\phi_0(s)}{4}P(x, s) + \phi_0(s)(1 - q)P(y = -2a, s). \quad (2.6)$$

Generalizing (2.5) to any node of the branches located between  $2a \leq y \leq (N - 2)a$ , we obtain the balance equation for the upper branches

$$P(y, s) = \phi_0(s) [qP(y - a, s) + (1 - q)P(y + a, s)]. \quad (2.7)$$

To determine the Laplace transform  $\phi(s)$  of the effective backbone node waiting time PDF, we need to determine  $P(y = a, s)$  and  $P(y = -a, s)$  in (2.3) in terms of  $P(x, t)$ , so that (2.3) can be cast in the form of (2.2). Given (2.5) and (2.6), this goal can be achieved if  $P(y = 2a, s)$  and  $P(y = -2a, s)$  can be related to  $P(y = a, s)$  and  $P(y = -a, s)$ . We proceed as follows. The solution of (2.7) reads

$$P(y, s) = A_1 \lambda_+^{y/a} + A_2 \lambda_-^{y/a}, \quad (2.8)$$

where

$$\lambda_{\pm} = \frac{1 \pm \sqrt{1 - 4q(1 - q)\phi_0^2(s)}}{2(1 - q)\phi_0(s)}. \quad (2.9)$$

To find expressions for the quantities  $A_1$  and  $A_2$ , whose dependence on  $x$  and  $s$  is not displayed, we apply (2.8) to the node  $y = 2a$ :

$$P(y = 2a, s) = A_1 \lambda_+^2 + A_2 \lambda_-^2. \quad (2.10)$$

On the other hand, setting  $y = 2a$  in (2.7), we find

$$P(y = 2a, s) = \phi_0(s) [qP(y = a, s) + (1 - q)\phi_0(s)P(y = 3a, s)], \quad (2.11)$$

or

$$P(y = 2a, s) - \phi_0(s)qP(y = a, s) = \phi_0(s)(1 - q)\phi_0(s)P(y = 3a, s). \quad (2.12)$$

Setting  $y = 3a$  in (2.8) we obtain

$$P(y = 2a, s) - q\phi_0(s)P(y = a, s) = \phi_0(s)(1 - q) [A_1 \lambda_+^3 + A_2 \lambda_-^3]. \quad (2.13)$$

Solving the system of equations (2.10) and (2.13) for the quantities  $A_1$  and  $A_2$ , we obtain

$$A_1 = \frac{P(y = 2a, s) - q\phi_0(s)P(y = a, s)}{\lambda_+^2 (\lambda_+ - \lambda_-) \phi_0(s)(1 - q)} - \frac{\lambda_- P(y = 2a, s)}{\lambda_+^2 (\lambda_+ - \lambda_-)}, \quad (2.14)$$

$$A_2 = \frac{-P(y = 2a, s) + q\phi_0(s)P(y = a, s)}{\lambda_-^2 (\lambda_+ - \lambda_-) \phi_0(s)(1 - q)} + \frac{\lambda_+ P(y = 2a, s)}{\lambda_-^2 (\lambda_+ - \lambda_-)}. \quad (2.15)$$

A special situation occurs at the end of the branches, where we have to impose reflecting boundary conditions, i.e.,

$$P(y = Na, s) = q\phi_0(s)P(y = (N - 1)a, s). \quad (2.16)$$

The node at  $y = (N - 1)a$  also needs a special balance equation (see Fig. 2),

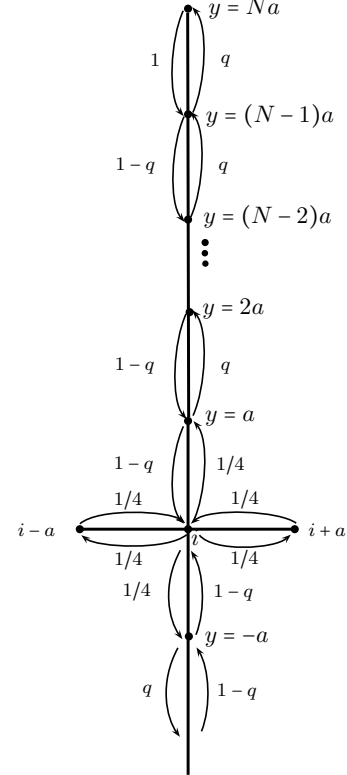


FIG. 2: Schematic representation of the possible jumps of a walker with the corresponding probabilities.

$$P(y = (N-1)a, s) = q\phi_0(s)P(y = (N-2)a, s) + \phi_0(s)P(y = Na, s). \quad (2.17)$$

Substituting  $y = (N-2)a$  into (2.7) and considering (2.16), we can write

$$P(y = (N-2)a, s) = h(\phi_0(s))P(y = (N-3)a, s), \quad (2.18)$$

where

$$h(\phi_0(s)) = \frac{q\phi_0(s) [1 - q\phi_0^2(s)]}{1 + q(q-2)\phi_0^2(s)}. \quad (2.19)$$

Substituting the solutions from (2.8), (2.14), and (2.15) into (2.18), we find

$$P(y = 2a, s) = G(q, \phi_0(s))P(y = a, s), \quad (2.20)$$

where

$$G(q, \phi_0(s)) = \frac{2q\phi_0(s)}{1 + \frac{1 + H(q, \phi_0(s))}{1 - H(q, \phi_0(s))} \sqrt{1 - 4q(1-q)\phi_0^2(s)}}, \quad (2.21)$$

$$H(q, \phi_0(s)) = \left( \frac{\lambda_-}{\lambda_+} \right)^{N-5} \frac{\lambda_- - h(\phi_0(s))}{\lambda_+ - h(\phi_0(s))}. \quad (2.22)$$

For the lower branch we obtain in a similar manner,

$$P(y = -2a, s) = G(q, \phi_0(s))P(y = -a, s). \quad (2.23)$$

We have achieved our goal of expressing  $P(y = 2a, s)$  and  $P(y = -2a, s)$  in terms of  $P(y = a, s)$  and  $P(y = -a, s)$ . Substituting (2.20) and (2.23) into (2.5) and (2.6) and using the resulting expressions in (2.3), we obtain an equation of the form (2.2) with

$$\phi(s) = \frac{\phi_0(s)}{2 - \frac{(1-q)\phi_0^2(s)}{1 - (1-q)\phi_0(s)G(q, \phi_0(s))}}. \quad (2.24)$$

The Laplace inversion of (2.24) yields  $\phi(\tau)$ , which incorporates the dynamics along the branches and can be understood as the effective waiting time PDF for a walker moving along the backbone only.

### III. STATISTICAL PROPERTIES

#### A. $N$ finite

If the local waiting time PDF  $\phi_0(\tau)$  has finite moments, its Laplace transform reads [2],  $\phi_0(s) \simeq 1 - s\bar{t}$ , in the large time limit  $s \rightarrow 0$ , where  $\bar{t}$  is the local mean waiting time at each node. Taking the limit  $s \rightarrow 0$  in (2.24), we obtain the waiting time PDF for the effective backbone dynamics,

$$\phi(s) \simeq (1 + s\langle t \rangle)^{-1}. \quad (3.1)$$

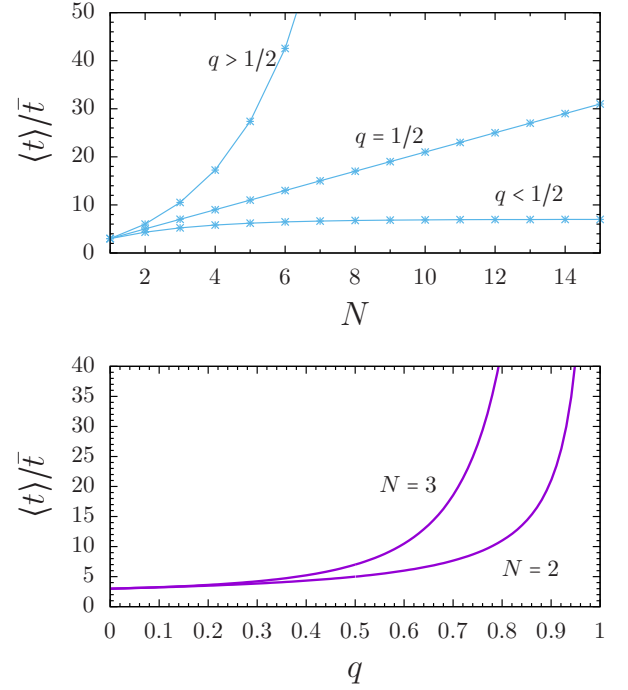


FIG. 3: Dimensionless mean waiting time of the effective backbone dynamics.

The mean waiting time  $\langle t \rangle$  is given by

$$\langle t \rangle = \frac{\bar{t}}{2q-1} [2(1-q)^{1-N}q^N + 4q - 3]. \quad (3.2)$$

In Fig. 3, we plot the effective mean waiting at a node of the backbone dynamics versus  $N$  and  $q$ . It shows that the mean waiting time  $\langle t \rangle$  is a monotonically increasing function of both  $q$  and  $N$ . If the random walk inside the branches is isotropic,  $q = 1/2$ , one obtains by the L'Hopital's rule from (3.2)

$$\lim_{q \rightarrow 1/2} \langle t \rangle = (1 + 2N)\bar{t}. \quad (3.3)$$

To determine the diffusion coefficient  $D$  for diffusion through the comb, we first calculate the MSD. Performing the Fourier-Laplace transform on (2.1), we obtain

$$P(k, s) = \frac{1 - \phi(s)}{s[1 - \Phi(k)\phi(s)]}. \quad (3.4)$$

The MSD in Laplace space reads (see, e.g., [2])

$$\langle x^2(s) \rangle = - \lim_{k \rightarrow 0} \frac{d^2 P(k, s)}{dk^2}. \quad (3.5)$$

As mentioned in Sec. II, we assume that the motion on the backbone is unbiased and that the walker only jumps to nearest neighbors. This implies that the kernel  $\Phi(x)$  is given  $\Phi(x) = \delta(x-a)/2 + \delta(x+a)/2$ , and we obtain from (3.5),

$$\langle x^2(s) \rangle = \frac{a^2}{s[\phi(s)^{-1} - 1]}. \quad (3.6)$$

in the large time limit. If the waiting time PDF  $\phi(t)$  possesses a finite first moment, (3.1) implies that the MSD along the backbone corresponds to normal diffusion  $\langle x^2(t) \rangle = 2Dt$ . The diffusion coefficient is given by

$$D = \frac{a^2}{2\langle t \rangle} = \frac{a^2}{2\bar{t}} \frac{2q-1}{2(1-q)^{1-N}q^N + 4q-3}. \quad (3.7)$$

In Fig. 4, we compare the results provided by (3.7) with numerical simulations.

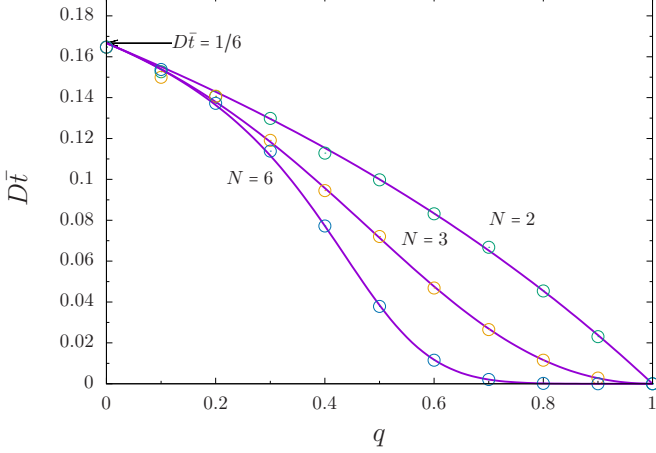


FIG. 4: Plot of the diffusion coefficient through the comb for  $N = 2$ ,  $N = 3$ , and  $N = 6$  versus  $q$ . Solid curves correspond to exact analytical results given by (3.7). Numerical simulations results are depicted with symbols.

As Fig. 3 demonstrates,  $\langle t \rangle$  increases monotonically with  $N$  for  $q < 1/2$  and saturates at  $(4q-3)/(2q-1)$  for  $N \rightarrow \infty$ . Consequently, the mean waiting time  $\langle t \rangle$  is finite for  $N \rightarrow \infty$ ; the overall diffusion along the backbone is normal. However, for  $q \geq 1/2$ , the mean waiting time  $\langle t \rangle$  increases without bound as  $N$  increases, and anomalous transport is expected for  $N \rightarrow \infty$ . In Fig. 5, we plot

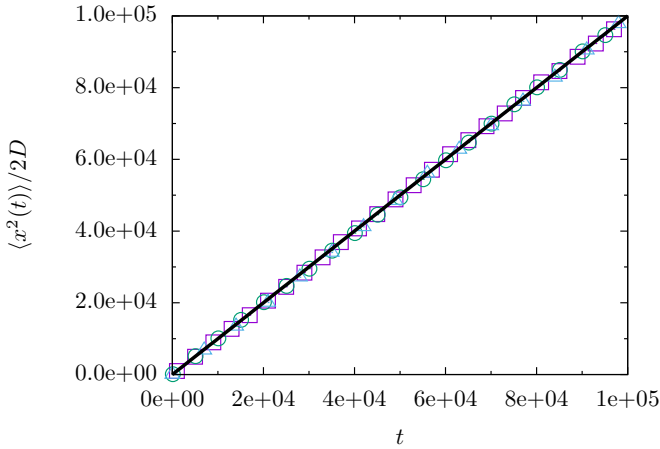


FIG. 5: MSD over  $2D$  for  $N$  fixed and three different values of  $q$ : 0.1, 0.25, 0.75

the MSD scaled by the diffusion coefficient. It illustrates

the result given by (3.7) for the MSD. The transport is diffusive for finite  $N$ , regardless  $q$  and the specific form of  $\phi_0(\tau)$ , as long as it has finite moments.

We consider now a an effective waiting time PDF with the large-time limit  $\phi_0(\tau) \sim \tau^{-1-\gamma}$ , with Laplace transform  $\phi_0(s) \simeq 1 - (s\tau_0)^\gamma$  and  $0 < \gamma < 1$ , which does not possess finite moments. Here  $\tau_0$  is a parameter with units of time. In this case, the waiting time PDF for the backbone dynamics is obtained by simply replacing  $s\bar{t}$  with  $(s\tau_0)^\gamma$ , i.e.,  $\phi(s) \simeq [1 + (s\tau_0)^\gamma \langle t \rangle / \tau_0]^{-1}$ . Substituting this result into (3.6), we find

$$\langle x^2(t) \rangle = \frac{a^2 \tau_0}{\langle t \rangle} \frac{(t/\tau_0)^\gamma}{\Gamma(1+\gamma)}, \quad (3.8)$$

for large  $t$ , where  $\langle t \rangle$  is given by (3.2), with  $\tau_0$  instead of  $\bar{t}$ . If the waiting time PDF  $\phi_0(\tau)$  at each node of the comb has a power-law tail, then the overall transport along the backbone is anomalous.

## B. $N \rightarrow \infty$

If the number of nodes of the branches goes to infinity, the mean time spent by the walker visiting a branch increases monotonically, see (3.2). However, this does not always results in anomalous transport along the overall structure as we show below.

For  $N \rightarrow \infty$ , the quotient  $(\lambda_-/\lambda_+)^N \rightarrow 0$  and also  $H \rightarrow 0$ . We obtain from (2.21),

$$G(q, \phi_0(s)) = \frac{2q\phi_0(s)}{1 + \sqrt{1 - 4q(1-q)\phi_0^2(s)}} \equiv \frac{2q\phi_0(s)}{1 + g(q)}, \quad (3.9)$$

where we define  $g(q) \equiv \sqrt{1 - 4q(1-q)\phi_0^2(s)}$  for convenience. Equation (2.24) reduces to

$$\phi(s) = \frac{\phi_0(s) [1 + g(q) - 2q(1-q)\phi_0^2(s)]}{2 - (1 + 3q - 4q^2)\phi_0^2(s) + [2 - (1-q)\phi_0^2(s)]g(q)}. \quad (3.10)$$

We take the limit  $s \rightarrow 0$  and consider first the case where  $\phi_0(\tau)$  has finite moments. Then  $\phi_0(s) \simeq 1 - s\bar{t}$ , as  $s \rightarrow 0$ . The square root  $g(q)$  in (3.10) reads

$$g(q) \simeq \begin{cases} 1 - 2q - \frac{4q(1-q)}{2q-1} s\bar{t}, & q < 1/2, \\ \sqrt{2s\bar{t}} - \frac{\sqrt{2}}{4} (s\bar{t})^{3/2}, & q = 1/2, \\ -1 + 2q + \frac{4q(1-q)}{2q-1} s\bar{t}, & q > 1/2, \end{cases} \quad (3.11)$$

and (3.10) implies that the waiting time PDF is given by

$$\phi(s) \simeq \begin{cases} \left(1 + \frac{4q-3}{2q-1} s\bar{t}\right)^{-1}, & q < 1/2, \\ \left(1 + \sqrt{2s\bar{t}}\right)^{-1}, & q = 1/2, \\ \left(\frac{3q-1}{q} + \frac{4q^2-3q+1}{(2q-1)q} s\bar{t}\right)^{-1}, & q > 1/2. \end{cases} \quad (3.12)$$

Substituting (3.12) into (3.6), we find for large  $t$ ,

$$\langle x^2(t) \rangle = \begin{cases} a^2 \frac{2q-1}{4q-3} \frac{t}{\bar{t}}, & q < 1/2, \\ a^2 \sqrt{\frac{2t}{\pi\bar{t}}}, & q = 1/2, \\ a^2 \frac{q}{2q-1} (1 - e^{-\alpha t}), & q > 1/2, \end{cases} \quad (3.13)$$

where the rate of saturation is

$$\alpha = \frac{(2q-1)^2}{(4q^2-3q+1)\bar{t}}. \quad (3.14)$$

In Fig 6 we compare these results with numerical simulations for  $N = 10^3$ . For  $q = 1/2$ , we obtain the well known result of subdiffusive transport with the MSD  $\sim \sqrt{t}$ . However, for  $q \neq 1/2$ , the side branches experience advection, and the transport is remarkably different. Namely, for  $q > 1/2$  the advection is away from the backbone along the branches,  $y \rightarrow \pm\infty$ . The walker is effectively trapped inside the branches, and stochastic localization (diffusion failure) occurs,  $\langle x^2(\infty) \rangle < \infty$ , [30]. For  $q < 1/2$ , the advection is towards the backbone. It enhances the backbone dynamics and normal diffusion takes place.

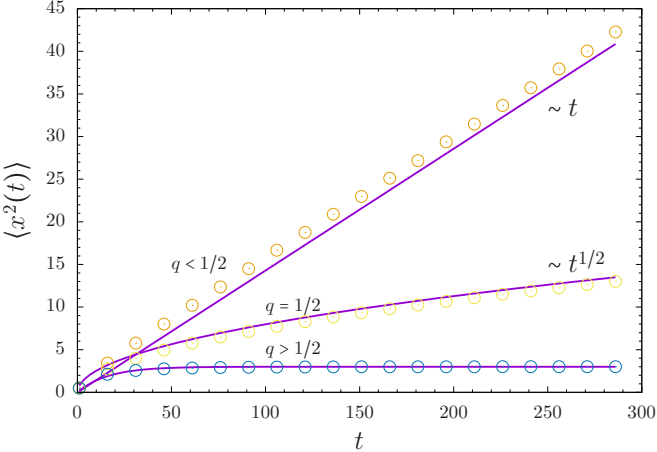


FIG. 6: MSD for three values of  $q$ , displaying three different behaviors. Solid curves correspond to the results given by (3.13). Symbols are the results of numerical simulations with  $N = 10^3$ .

Consider now the case where the local waiting time PDF is  $\phi_0(\tau) \sim \tau^{-1-\gamma}$ , i.e.,  $\phi_0(s) \simeq 1 - (s\tau_0)^\gamma$  with

$0 < \gamma < 1$ , as  $s \rightarrow 0$ . The MSD in this case can be obtained straightforwardly by replacing  $s\bar{t}$  with  $(s\tau_0)^\gamma$  in (3.12). For large times it reads

$$\langle x^2(t) \rangle = \begin{cases} \frac{a^2}{\Gamma(1+\gamma)} \frac{2q-1}{4q-3} \left(\frac{t}{\tau_0}\right)^\gamma, & q < 1/2, \\ \frac{a^2}{\sqrt{2}\Gamma(1+\gamma/2)} \left(\frac{t}{\tau_0}\right)^{\gamma/2}, & q = 1/2, \\ \frac{a^2 q (2q-1)}{\tau_0^\gamma (4q^2-3q+1)} \mu(t/\tau_0), & q > 1/2, \end{cases} \quad (3.15)$$

where

$$\mu(t/\tau_0) = (t/\tau_0)^\gamma E_{\gamma, \gamma+1} \left[ - \left(\frac{t}{\tau_0}\right)^\gamma \frac{(2q-1)^2}{4q^2-3q+1} \right] \quad (3.16)$$

is expressed in terms of the generalized Mittag-Leffler function  $E_{\alpha, \beta}(z)$ . We use the following property of integration of the Mittag-Leffler function [31],

$$\int_0^t E_{\alpha, \beta}(bz^\alpha) z^{\beta-1} dz = t^\beta E_{\alpha, \beta+1}(bt^\alpha). \quad (3.17)$$

Subdiffusion in the branches results in backbone subdiffusion for  $q \leq 1/2$ . For advection away from the backbone,  $q > 1/2$ , we again find stochastic localization. For  $t/\tau_0 \gg 1$ ,  $E_{\alpha, \beta}(-at^\alpha) \sim t^{-\alpha}/\Gamma(\beta - \alpha)$  [32], and consequently  $\mu(t/\tau_0)$  approaches a finite value as  $t \rightarrow \infty$ .

#### IV. CONCLUSION

We have developed a mesoscopic equation for a random walk on a regular comb structure given by (2.2) and (2.24). The random walk along the branches consists of, possibly biased, jumps to the nearest node, while waiting at each node for a random time  $\tau$  distributed according to the PDF  $\phi_0(\tau)$  before proceeding with the next jump. The overall dynamics along the branches has been reduced to an effective waiting time PDF, given by (2.24), for motion solely along the backbone. We have obtained statistical properties, such as the effective mean waiting time,  $\langle t \rangle$  for the backbone nodes, and the diffusion coefficient,  $D$ , of the overall structure for the case where the number of nodes  $N$  of the branches is finite or infinite. If  $N$  is finite and  $\phi_0(\tau)$  has finite moments, both  $\langle t \rangle$  and  $D$  are derived exactly in terms of the bias probability  $q$ , the number of nodes  $N$  on the branch, and the mean waiting time probability at each node. In this case the transport is always normal diffusion. If  $\phi_0(\tau) \sim \tau^{-1-\gamma}$  for large time, it does not possess finite moments and the MSD of the random walker behaves like  $t^\gamma$ . If  $N$  is infinite, the value of  $q$  is decisive. If  $\phi_0(\tau)$  has finite moments, the diffusion regime is normal if  $q < 1/2$ , while the MSD behaves like  $t^{1/2}$  for  $q = 1/2$ . If  $q > 1/2$ , the MSD approaches a constant finite value for large time, corresponding to stochastic localization (diffusion failure). If  $\phi_0(\tau) \sim \tau^{-1-\gamma}$  for large time, the MSD behaves like  $t^\gamma$ .

for  $q < 1/2$  and like  $t^{\gamma/2}$  for  $q = 1/2$ . Again, stochastic localization occurs for  $q > 1/2$ . In summary, if the bias probability of moving away from the backbone is  $q > 1/2$ , then stochastic localization occurs, regardless of the other characteristic parameters related to the random walk on the branches.

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